Solution of Math2060's midterm

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Contents

1 Question 1

Using Mean Value Theorem to show that

(a) $|\sin x - \sin y| \le |x - y|$ for all $x, y \in \mathbb{R}$.

(b) $\frac{x-1}{x} < \ln x < x - 1$ for all $x > 1$.

Proof. (a) Since $\sin x$ is differentiable on $\mathbb{R}$ and $(\sin x)' = \cos x$, by mean value theorem,

$$\sin x - \sin y = \cos c \cdot (x - y)$$

for some constant c between x and y. Notice that $|\cos x| \le 1$. This completes the proof.

(b) Since $\ln x$ is differentiable on $(0, \infty)$ and $(\ln x)' = \frac{1}{x}$, by mean value theorem,

$$\ln x - \ln 1 = \frac{1}{c} \cdot (x - 1)$$

for some constant $1 < c < x$.

Thus, $\frac{x-1}{x} < \ln x = \frac{x-1}{c} < \frac{x-1}{1}$. □

2 Question 2

Let $f(x) = \sin \frac{\pi}{x}$ for $x \in (0, 2)$. FInd the 2nd-order Taylor polynomial $P_2(x)$ for f at $x_0 = 1$. and the remainder $R_2(x)$ in Lagrange form.

Proof.

$$\begin{cases} f(x) = \sin \frac{\pi}{x} \\ f'(x) = -\frac{\pi}{x^2} \cos \frac{\pi}{x} \\ f''(x) = -\frac{\pi^2}{x^4} \sin \frac{\pi}{x} + \frac{2\pi}{x^3} \cos \frac{\pi}{x} \\ f^{(3)}(x) = (\frac{\pi^3}{x^6} - \frac{6\pi}{x^4}) \cos \frac{\pi}{x} + \frac{6\pi^2}{x^5} \sin \frac{\pi}{x} \end{cases}$$

Thus, $f(1) = 0$, $f'(1) = \pi$ and $f''(1) = -2\pi$.

Thus, $P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 = \pi(x-1) - \pi(x-1)^2$

and $R_2(x) = \frac{1}{6}\left[(\frac{\pi^3}{c^6} - \frac{6\pi}{c^4}) \cos \frac{\pi}{c} + \frac{6\pi^2}{c^5} \sin \frac{\pi}{c}\right](x-1)^3$ for some constant c between 1 and x. □

3 Question 3

Let $f : [-1, 1] \to \mathbb{R}$ be a function such that $f'(x)$ exists and continuous on $[-1, 1]$ and $f''(x)$ exists on $(-1, 1)$. If $g(x) = x(x^2 - 1)f(x)$, show that there is a point $c \in (-1, 1)$ such that $g''(c) = 0$.

Proof. Since $g(0) = 0 = g(-1)$ and g' exists on $(-1, 0)$, by mean value theorem, there exists a constant $c_1 \in (-1, 0)$ such that $g'(c_1) = 0$. By the same reason, there exists a constant $c_2 \in (0, 1)$ such that $g'(c_2) = 0$. Since g'' exists on $(c_1, c_2) \subset (-1, 1)$, by mean value theorem, $g''(c) = 0$ for some constant $c \in (c_1, c_2)$. □

4 Question 4

Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function such that

$$\lim_{x\to\infty} f(x) + f'(x) = L$$

where $L \in \mathbb{R}$. Show that

$$\lim_{x\to\infty} f(x) = L.$$

Proof. Notice that $f(x) = \frac{e^x f}{e^x}$ for all $x \in \mathbb{R}$ and $e^x f$, e^x differentiable on $(0, \infty)$, as well as $\lim_{x\to\infty} e^x = +\infty$.

Using the L'Hôpital's rule,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^x f}{e^x} = \lim_{x\to\infty} \frac{(e^x f)'}{(e^x)'} = \lim_{x\to\infty} f(x) + f'(x) = L$$

□

5 Question 5

Let $f : [-1, 1] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} x^3 \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

(a) Show $f'(0)$ exists and find its value.

(b) Show $\lim_{h\to 0} \frac{h(0+h)-2f(0)+f(0-h)}{h^2}$ exists and find its value.

(c) Does $f''(0)$ exists? Justify your answer.

Proof. (a)

$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} h^2 \sin(\frac{1}{h}) = 0.$$

(b)

$$\lim_{h\to 0} \frac{h(0+h)-2f(0)+f(0-h)}{h^2} = \lim_{h\to 0} 2h\sin(\frac{1}{h}) = 0.$$

(c) No.

$$f'(x) = \begin{cases} 3x^2\sin(\frac{1}{x}) - x\cos(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

$$f''(0) = \lim_{h\to 0} \frac{f'(h)-f'(0)}{h} = \lim_{h\to 0} 3h\sin(\frac{1}{h}) - \cos(\frac{1}{h}).$$

The limit doesn't exist. □

6 Question 6

Give definition of $f : [a,b] \to \mathbb{R}$ being Riemann integrable on $[a,b]$.

Proof. (a) There exists a constant $L \in \mathbb{R}$ such that for any $\epsilon > 0$, there is a constant $\delta > 0$ satisfying

$$|S(f,\dot{P}) - L| < \epsilon$$

whenever $\|\dot{P}\| \leq \delta$.

(b) Prove it by definition. Fix $\epsilon > 0$, let $E_\epsilon = \{x : G(x) > \frac{\epsilon}{4}\}$. Then E_ϵ has only finite points, denoting its number as N_ϵ. Let $\delta = \frac{\epsilon}{4N_\epsilon}$. Let $\dot{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ be the partition such that $\|\dot{P}\| \leq \delta$. Then

$$\begin{aligned} 0 \leq S(G;\dot{P}) =& \sum_{t_i\in E_\epsilon} G(t_i)(x_i - x_{i-1}) + \sum_{t_i\notin E_\epsilon} G(t_i)(x_i - x_{i-1}) \\ \leq& 2N_\epsilon\delta + 2\frac{\epsilon}{4} \leq \epsilon. \end{aligned}$$

Thus, G is Riemann integrable and its value is 0. □